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Special Functions in Mathematics

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ABSTRACT: A special function is a function (usually named after an early investigator of its properties) having a particular use in mathematical physics or some other branch of mathematics. Prominent examples include the gamma function, hypergeometric function, Whittaker function, and Meijer G-function. Special function, any of a class of mathematical functions that arise in the solution of various classical problems of physics. These problems generally involve the flow of electromagnetic, acoustic, or thermal energy. Different scientists might not completely agree on which functions are to be included among the special functions, although there would certainly be very substantial overlap.

At first glance, the physical problems mentioned above seem to be very limited in scope. From a mathematical point of view, however, different representations have to be sought, depending on the configuration of the physical system for which these problems are to be solved. For example, in studying propagation of heat in a metallic bar, one could consider a bar with a rectangular cross section, a round cross section, an elliptical cross section, or even more-complicated cross sections; the bar might be straight or curved. Every one of these situations, while dealing with the same type of physical problem, leads to somewhat different mathematical equations.

KEYWORDS-physics,mathematical,special,functions,hypergeometric,propagation,acoustic

I. INTRODUCTION

The equations to be solved are partial differential equations. To apprehend how these equations come about, one can consider a straight rod along which there is a uniform flow of heat. Let $u(x, t)$ denote the temperature of the rod at time t and location x, and let $q(x, t)$ denote the rate of heat flow. The expression $\partial q/\partial x$ denotes the rate at which the rate of heat flow changes per unit length and therefore measures the rate at which heat is accumulating at a given point x at time t. If heat is accumulating, the temperature at that point is rising, and the rate is denoted by $\partial u/\partial t$. The principle of conservation of energy leads to $\partial q/\partial x = k(\partial u/\partial t)$, where k is the specific heat of the rod. This means that the rate at which heat is accumulating at a point is proportional to the rate at which the temperature is increasing. A second relationship between q and u is obtained from Newton's law of cooling, which states that $q = K(\partial u/\partial x)$. The latter is a mathematical way of asserting that the steeper the temperature gradient (the rate of change of temperature per unit length), the higher the rate of heat flow. Elimination of q between these equations leads to $\partial^2 u/\partial x^2 = (k/K)(\partial u/\partial t)$, the partial differential equation for one-dimensional heat flow.[1,2,3]

The partial differential equation for heat flow in three dimensions takes the form $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (k/K)(\partial u/\partial t)$; the latter equation is often written $\nabla^2 u = (k/K)(\partial u/\partial t)$, where the symbol ∇, called del or nabla, is known as the Laplace operator. ∇ also enters the partial differential equation dealing with wave-propagation problems, which has the form $\nabla^2 u = (1/c^2)(\partial^2 u/\partial t^2)$, where c is the speed at which the wave propagates.[5,7,8]

Partial differential equations are harder to solve than ordinary differential equations, but the partial differential equations associated with wave propagation and heat flow can be reduced to a system of ordinary differential equations through a process known as separation of variables. These ordinary differential equations depend on the choice of coordinate system, which in turn is influenced by the physical configuration of the problem. The solutions of these ordinary differential equations form the majority of the special functions of mathematical physics.

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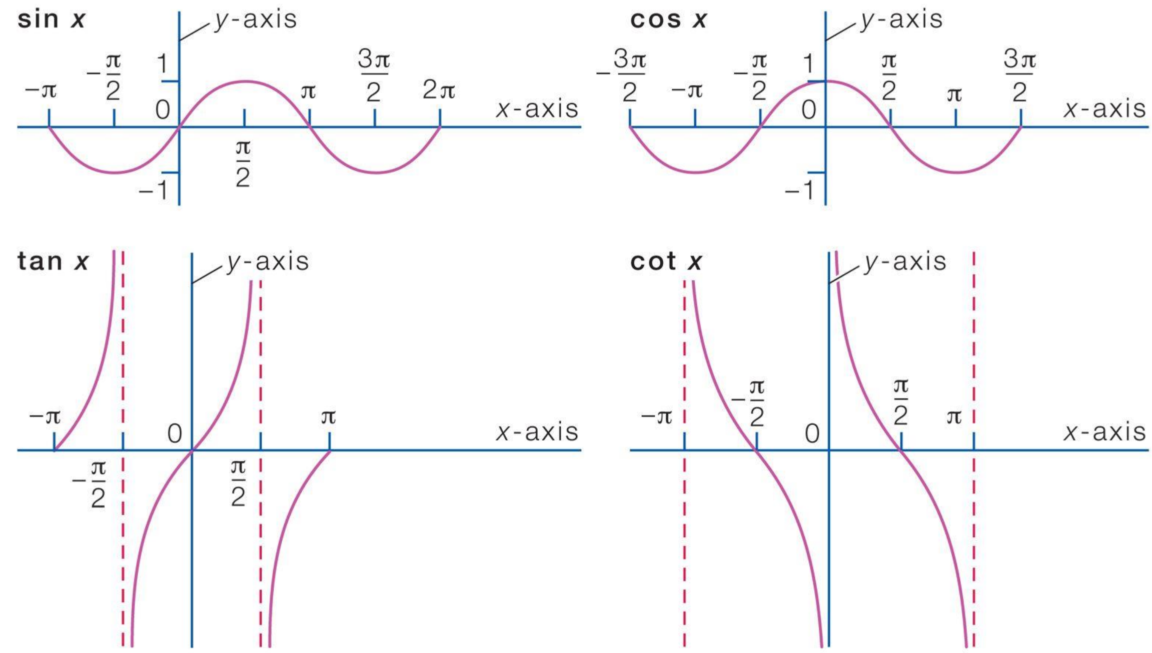

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For example, in solving the equations of heat flow or wave propagation in cylindrical coordinates, the method of separation of variables leads to Bessel's differential equation, a solution of which is the Bessel function, denoted by $J_n(x)$.[9,10,11]

Among the many other special functions that satisfy second-order differential equations are the spherical harmonics (of which the Legendre polynomials are a special case), the Tchebychev polynomials, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Whittaker functions, and the parabolic cylinder functions. As with the Bessel functions, one can study their infinite series, recursion formulas, generating functions, asymptotic series, integral representations, and other properties. Attempts have been made to unify this rich topic, but not one has been completely successful. In spite of the many similarities among these functions, each has some unique properties that must be studied separately. [12,13,14]

$$(1) \qquad x^2y'' + xy' + \left(-v^2 + x^2\right)y = \frac{4\left(\frac{x}{2}\right)^{v+1}}{\sqrt{\pi}\,\Gamma\left(v+\frac{1}{2}\right)}$$

$$(2) \qquad x^2y'' + xy' - \left(v^2 + x^2\right)y = \frac{4\left(\frac{x}{2}\right)^{v+1}}{\sqrt{\pi}\,\Gamma\left(v+\frac{1}{2}\right)}$$

But some relationships can be developed by introducing yet another special function, the hypergeometric function, which satisfies the differential equation $z(1-z)\, d^2y/dx^2 + [c-(a+b+1)z]\, dy/dx - aby = 0$. Some of the special functions can be expressed in terms of the hypergeometric function.

While it is true, both historically and practically, that the special functions and their applications arise primarily in mathematical physics, they do have many other uses in both pure and applied mathematics. Bessel functions are useful in solving certain types of random-walk problems. They also find application in the theory of numbers. The hypergeometric functions are useful in constructing so-called conformal mappings of polygonal regions whose sides are circular arcs.

Gamma function

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx$$

$$\Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n+1) = n!$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Beta function

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\beta(m,n) = \beta(n,m)$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

II. DISCUSSION

Special functions have also been traditionally significant in both algebraic geometry and integrable systems.

$$\Theta_{10} = \frac{1}{2}\sqrt{6}\cos\theta \qquad \Theta_{30} = \frac{3}{4}\sqrt{14}\left[\frac{5}{3}\cos^3\theta - \cos\theta\right]$$

$$\Theta_{11} = \frac{1}{2}\sqrt{3}\sin\theta \qquad \Theta_{31} = \frac{1}{8}\sqrt{42}\sin\theta\left(5\cos^2\theta - 1\right)$$

$$\Theta_{20} = \frac{1}{4}\sqrt{10}(3\cos^2\theta - 1) \qquad \Theta_{32} = \frac{1}{4}\sqrt{105}\sin^2\theta\cos\theta$$

$$\Theta_{21} = \frac{1}{2}\sqrt{15}\sin\theta\cos\theta \qquad \Theta_{33} = \frac{1}{8}\sqrt{70}\sin^3\theta$$

$$\Theta_{22} = \frac{1}{4}\sqrt{15}\sin^2\theta$$

Within the examples presented, elliptic functions gave rise to surprisingly sophisticated theories. The 1-wave solution encountered in the introduction, $u = 2\wp$ + const. in the limit when one or both periods of the Weierstrass function go to zero, becomes exponential or rational, respectively. The higher-genus analogs give rise to solitons, or rational solutions. On the other hand, the KP solutions which are doubly periodic in the x variable ("elliptic solitons") were classified by Krichever (cf. Dubrovin et al. (2001)), as forming an ACI Hamiltonian system ("elliptic Calogero–Moser"), which, 25 years later, is still generating important work, with Hamiltonian.

The associated spectral curves have been classified in moduli by Treibich and Verdier (cf. Treibich (2001)); Krichever produced a two-field model as well as a universal Poisson structure for the system; Donagi and Markman (1996) realized it as a generalized Hitchin system.

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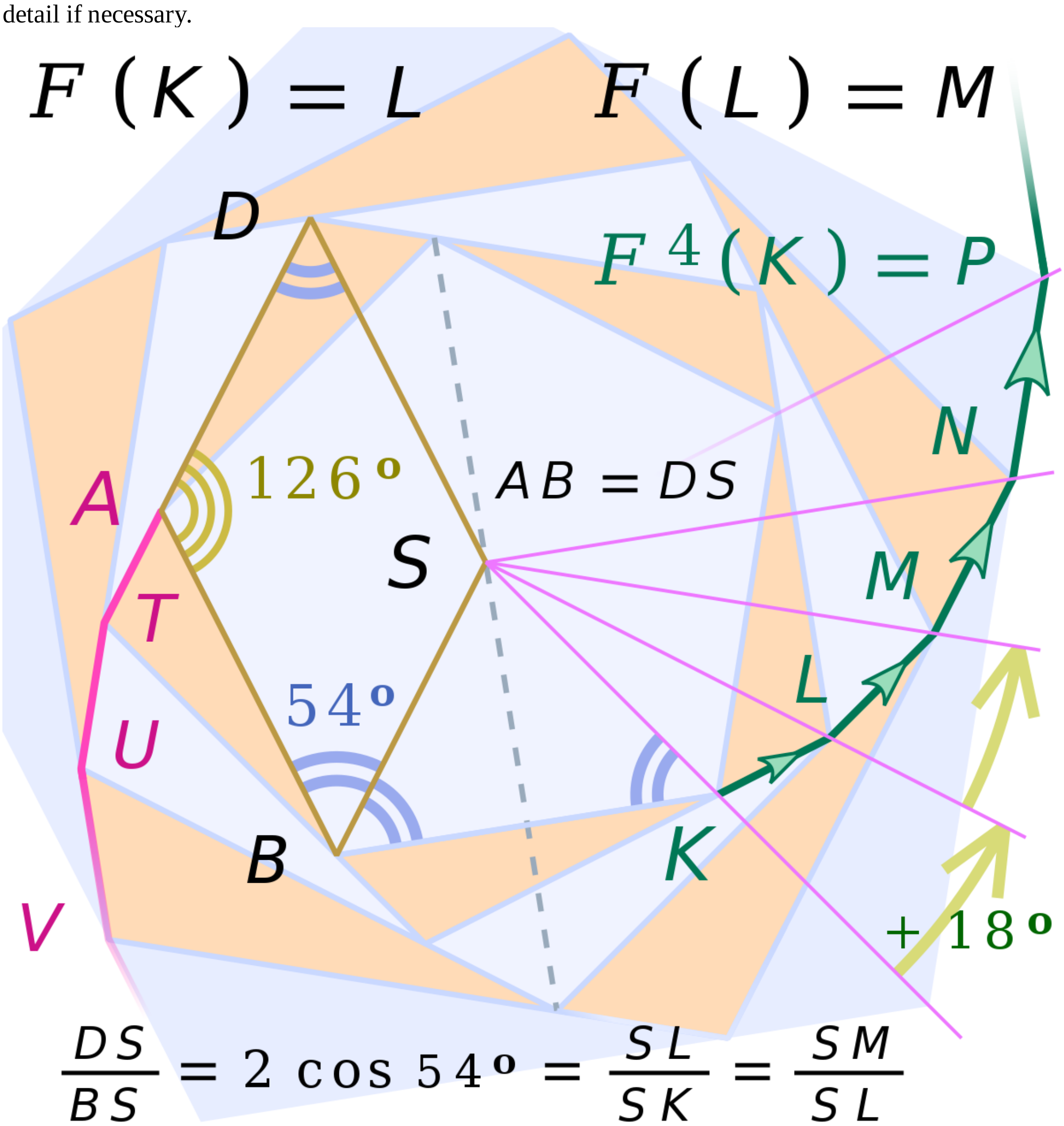
More classically, elliptic potentials were the subject of much study, in particular by Lamé and Hermite in the nineteenth century and Ince in the twentieth; a sample result due to Ince makes one feel like Alice in Wonderland, who “knelt down and looked along the passage into the loveliest garden you ever saw”: the Lamé operator $L = -\partial^2 + a(a+1)\wp(x - x_0)$ with real, smooth potential is finite gap (namely, almost all the periodic eigenvalues are double) iff ∈ (if a is positive the number of gaps is a). A generalization to several variables (due to Chalykh and Veselov), Roughly speaking, this means that the centralizer of L contains n operators with functionally independent symbols, where n is the number of variables.[8,9,10]

A number of special functions have become important in physics because they arise in frequently encountered situations. Identifying a one-dimensional (1-D) integral as one yielding a special function is almost as good as a straight-out evaluation, in part because it prevents the waste of time that otherwise might be spent trying to carry out the integration. But of perhaps more importance, it connects the integral to the full body of knowledge regarding its properties and evaluation. It is not necessary for every physicist to know everything about all known special functions, but it is desirable to have an overview permitting the recognition of special functions which can then be studied in more detail if necessary.



It is common for a special function to be defined in terms of an integral over the range for which that integral converges, but to have its definition extended to a larger domain by analytic continuation in the complex plane

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Special functions, being natural generalizations of the elementary functions, have their origin in the solution of partial differential equations satisfying some set of conditions. Special functions can be defined in a variety of ways. Many special functions of a complex variable can be defined by means of either a series or an appropriate integral. Special functions like Bessel functions, Whittaker functions, Gauss hypergeometric function and the polynomials that go by the names of Jacobi, Legendre, Laguerre, Hermite, etc., have been continuously developed. Also, sequences of polynomials play a vital role in applied mathematics. Two important classes of polynomial sequences are the Sheffer and Appell sequences. The Appell and Sheffer polynomial sequences occur in different applications in many different branches of mathematics, theoretical physics, approximation theory, and other fields.[11,12]

This special issue focuses on the applications of the special functions and polynomials to various areas of mathematics. Thorough knowledge of special functions is required in modern engineering and physical science applications. These functions typically arise in such applications as communications systems, statistical probability distribution, electro-optics, nonlinear wave propagation, electromagnetic theory, potential theory, electric circuit theory, and quantum mechanics.

III. RESULTS

The elementary functions that appear in the first few semesters of calculus – powers of x, ln, sin, cos, exp, etc. are not enough to deal with the many problems that arise in science and engineering. Special function is a term loosely applied to additional functions that arise frequently in applications. We will discuss three of them here: Bessel functions, the gamma function, and Legendre polynomials. Bessel functions come up in problems with circular or spherical symmetry. As an example, we will look at the problem of heat flow in a circular plate. This problem is the same as the one for diffusion in a plate. By adjusting our units of length and time, we may assume that the plate has a unit radius.
Recall that in discussing the heat flow problem in section 1.1, we arrived at an eigenvalue problem in which we needed to find a function R(r) that satisfies (2), the condition R(1) = 0, and is continuous at r = 0. We then reduced that problem to solving Bessel's equation (3), with y(x) = R(r) and x = √ λr. Now, what we have seen is that y = Jn(x), which is defined by the series in (12), solves (3). Also, for any integer n ≥ 0, the solution behaves like x n near x = 0, so it is continuous there. We thus have that R(r) = Jn(√ λr) satisfies two of three conditions on R. The last one is that R(1) = 0, and it will be satisfied if Jn(√ λ) = 0 – i.e., if √ λ > 0 is a positive zero of Jn. (For all n > 0, Jn(0) = 0.)[13]
The gamma function is an extension of factorials to other numbers besides nonnegative integers. It will allow us to make sense out of expressions like (3/2)!, (−1/3)!, and so on.
Legendre polynomials come up in connection with three dimensional problems having spherical symmetry. The simplest situation that gives rise to them is solving for the steady-state temperature u(r,ϕ,ϕ) in a sphere of radius 1. (We will use the convention that ϕ is the colatitude, which angle measured off of the z-axis, and θ is the azimuthal angle, effectively the longitude.)

ERROR FUNCTIONS

- In mathematics, the error function (also called the Gauss error function) is a special function that occurs in probability, statistics, and partial differential equations describing diffusion. It is defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t.$$

- The **complementary error function**, denoted *erfc*, is defined as

$$\begin{aligned}\operatorname{erfc}(x) &= 1 - \operatorname{erf}(x) \\ &= \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\,\mathrm{d}t \\ &= e^{-x^2} \operatorname{erfcx}(x),\end{aligned}$$

IV. CONCLUSIONS

Special functions enable us to formulate a scientific problem by reduction such that a new, more concrete problem can be attacked within a well-structured framework, usually in the context of differential equations. A good understanding of special functions provides the capacity to recognize the causality between the abstractness of the mathematical concept and both the impact on and cross-sectional importance to the scientific reality. The special functions to be discussed in this monograph vary greatly, depending on the measurement parameters examined (gravitation, electric and magnetic fields, deformation, climate observables, fluid flow, etc.) and on the respective field characteristic (potential field, diffusion field, wave field). The differential equation under consideration determines the type of special functions that are needed in the desired reduction process. Each chapter closes with exercises that reflect significant topics, mostly in computational applications. As a result, readers are not only directly confronted with the specific contents of each chapter, but also with additional knowledge on mathematical fields of research, where special functions are essential to application.[14]

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